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Supersymmetric partners of the trigonometric Pöschl–Teller potentials

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Abstract
The first- and second-order supersymmetry transformations are used to generate Hamiltonians with known spectra departing from the trigonometric Pöschl–Teller potentials. The several possibilities of manipulating the initial spectrum are fully explored, and it is shown how to modify one or two levels, or even to leave the spectrum unaffected. The behaviour of the new potentials at the boundaries of the domain is studied.

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1. Introduction

There is a growing interest nowadays in the design of systems whose Hamiltonians have prescribed energy spectra, and the simplest technique to achieve this goal is to use supersymmetric quantum mechanics (SUSY QM) [1]. In this procedure, departing from an initial solvable Hamiltonian H a new solvable one, $\widetilde{H}$, can be constructed with a slightly modified spectrum, by using a finite-order differential intertwining operator [2–27]. The ingredients to implement these transformations are seed solutions of the initial stationary Schrödinger equation associated with factorization energies which do not coincide in general with the eigenvalues of H. By iterating appropriately this method as many times as needed, one could construct Hamiltonians whose spectra are arbitrarily close to any desired one.

In the case that the intertwining operator is of first order the procedure can be implemented by using as the seed one Schrödinger solution for which factorization energy is less than or equal to the ground-state energy of H [2–16]. In order to surpass successfully this restriction, one needs to use interwining operators of at least second order [17–27]. The resulting second-order SUSY QM offers several interesting possibilities of spectral manipulation [15, 18, 19]: (i) two new levels can be placed between a pair of neighbouring physical ones E_{i-1}, E_i of H; (ii) one new energy can be created at an arbitrary position; (iii) one level can be moved; (iv) there is not modification of the initial spectrum; (v) one physical energy can be deleted; (vi) two neighbouring physical levels can be deleted.

The SUSY techniques have been extensively applied to several interesting examples for which the x-domain is the full real line (e.g. the harmonic oscillator) or the positive semi-axis (e.g. the radial oscillator or the Coulomb problem). In order to complete the scheme, it is important to apply them to cases where the x-domain is a finite interval, let us say $[x_l, x_r]$. An example of this kind, to be explored in detail in this paper, is the trigonometric Pöschl–Teller potential [16, 27–29]. This is closely related to several potentials widely used in molecular and solid state physics [28]. Since the SUSY transformations modify slightly the initial spectrum, it turns out that a lot of new potentials are available to be used as models in physical applications.

In the following section we will survey quickly the kth order SUSY QM, with special emphasis placed on the first- and second-order cases [15]. In section 3, we will build up the first- and second-order SUSY partners of the trigonometric Pöschl–Teller potential. In section 4, we will finish the paper with our conclusions.

2. Supersymmetric quantum mechanics

The study of systems ruled by the supersymmetry algebra with two generators,

$$[Q_i, H_{\rm ss}] = 0, \qquad \{Q_i, Q_j\} = \delta_{ij} H_{\rm ss}, \qquad i, j = 1, 2, \tag{2.1}$$

realized as

$$Q_1 = \frac{Q + Q^\dagger}{\sqrt{2}}, \qquad Q_2 = \frac{Q^\dagger - Q}{{\rm i}\sqrt{2}}, \tag{2.2}$$

$$Q = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & B^\dagger \\ 0 & 0 \end{pmatrix}, \qquad H_{\rm ss} = \begin{pmatrix} B^\dagger B & 0 \\ 0 & BB^\dagger \end{pmatrix}, \tag{2.3}$$

where $B^\dagger$ is a kth order differential operator intertwining two Schrödinger Hamiltonians H, $\widetilde{H}$ as

$$\widetilde{H} B^\dagger = B^\dagger H, \tag{2.4}$$

$$H = -\frac{1}{2}\frac{{\rm d}^2}{{\rm d}x^2} + V(x), \qquad \widetilde{H} = -\frac{1}{2}\frac{{\rm d}^2}{{\rm d}x^2} + \widetilde{V}(x), \tag{2.5}$$

is called *kth order supersymmetric quantum mechanics*. In this approach there is a relationship between the supersymmetric 'Hamiltonian' $H_{\rm ss}$ and the *physical* one $H^{\rm p} = {\rm diag}\{\widetilde{H}, H\}$ of polynomial type:

$$H_{\rm ss} = \prod_{i=1}^{k} (H^{\rm p} - \epsilon_i). \tag{2.6}$$

If one assumes that $V(x)$ is a given solvable potential with normalized eigenfunctions $\psi_n(x)$ and eigenvalues $E_n, n = 0, 1, \ldots$, equations (2.4) and (2.6) ensure that for any $\psi_n(x)$ such that $B^\dagger \psi_n(x) \neq 0$ it turns out that

$$\widetilde{\psi}_n(x) = \frac{B^\dagger \psi_n(x)}{\sqrt{(E_n - \epsilon_1)\cdots(E_n - \epsilon_k)}} \tag{2.7}$$

is a normalized eigenfunction of $\widetilde{H}$ with eigenvalue E_n. In general, the set $\{\widetilde{\psi}_n(x), n = 0, 1, \ldots\}$ is not complete, since there can exist eigenstates $\widetilde{\psi}_{\epsilon_i}(x)$ of $\widetilde{H}$ with eigenvalues ϵ_i

belonging as well to the kernel of B. By adding them to the previous set, the maximal set of eigenfunctions of $\widetilde{H}$ is thus given by

$$\{\widetilde{\psi}_{\epsilon_i}(x), \widetilde{\psi}_n(x), i = 1, \ldots, k, n = 0, 1, \ldots\}. \tag{2.8}$$

The corresponding eigenvalues are $\{\epsilon_i, E_n, i = 1, \ldots, k, n = 0, 1, \ldots\}$.

In the maximal situation, the potential $\widetilde{V}(x)$ as well as the complete set of eigenfunctions of $\widetilde{H}$ are determined once the seed eigenfunctions $u_i(x)$ of H (which not necessarily are physical) with eigenvalues ϵ_i, $i = 1, \ldots, k$ are supplied. In particular, $\widetilde{V}(x)$ reads

$$\widetilde{V}(x) = V(x) - \{\ln[W(u_1, \ldots, u_k)]\}'', \tag{2.9}$$

$W(u_1, \ldots, u_k)$ denoting the Wronskian of the seeds $u_1(x), \ldots, u_k(x)$. Let us illustrate the procedure more explicitly by means of the first- and second-order cases.

2.1. First-order supersymmetric quantum mechanics

Let us suppose that the intertwining operator is of first order

$$B^\dagger = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + \alpha(x)\right], \tag{2.10}$$

where the *superpotential* $\alpha(x)$ is to be determined. The use of equation (2.4) leads to

$$\widetilde{V}(x) = V(x) - \alpha'(x), \tag{2.11}$$

$$\alpha'(x) + \alpha^2(x) = 2[V(x) - \epsilon], \tag{2.12}$$

i.e., $\alpha(x)$ must satisfy the Riccati equation (2.12). On the other hand, if a function $u(x)$ such that $\alpha(x) = [\ln u(x)]'$ is employed, equations (2.11) and (2.12) become

$$\widetilde{V}(x) = V(x) - [\ln u(x)]'', \tag{2.13}$$

$$Hu(x) = \epsilon u(x), \tag{2.14}$$

namely, $u(x)$ obeys the initial stationary Schrödinger equation associated with ϵ.

Let us take now a solution $\alpha(x)$ (or $u(x)$) to the Riccati (or Schrödinger) equation (2.12) (or (2.14)) for a fixed factorization energy $\epsilon \leqslant E_0$, where E_0 is the ground-state energy of H. Thus, equations (2.11) and (2.13) indicate that the potential $\widetilde{V}(x)$ is determined completely, with a maximal set of normalized eigenfunctions $\{\widetilde{\psi}_\epsilon(x), \widetilde{\psi}_n(x)\}$ given by

$$\widetilde{\psi}_\epsilon(x) \propto \exp\left[-\int_0^x \alpha(y)\,\mathrm{d}y\right] = \frac{1}{u(x)}, \qquad \widetilde{\psi}_n(x) = \frac{B^\dagger\psi_n(x)}{\sqrt{E_n - \epsilon}}. \tag{2.15}$$

The corresponding eigenvalues are $\{\epsilon, E_n, n = 0, 1, \ldots\}$. Let us point out that the aim of the restriction $\epsilon \leqslant E_0$ is to avoid that singularities appear in $\alpha(x)$, $\widetilde{V}(x)$ and also in the $\widetilde{\psi}_\epsilon(x)$, $\widetilde{\psi}_n(x)$ of (2.15). Indeed, if $\epsilon > E_0$ the seed solution $u(x)$ will always have nodes in the x-domain of H and thus $\alpha(x)$ would have singularities at those points. If $\epsilon \leqslant E_0$, however, $u(x)$ can have at most one zero. In particular, there is a subset of nodeless u-functions in the two-dimensional space of solutions associated with $\epsilon \leqslant E_0$, which will be used in the following for implementing the non-singular first-order SUSY transformations.

2.2. Second-order supersymmetric quantum mechanics

Now, let the intertwining operator be of second order

$$B^\dagger = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \eta(x)\frac{\mathrm{d}}{\mathrm{d}x} + \gamma(x)\right), \tag{2.16}$$

where $\eta(x), \gamma(x)$ are to be determined. Equation (2.4) leads to a set of equations relating $V(x)$, $\widetilde{V}(x)$, $\eta(x)$, $\gamma(x)$ and their derivatives which, after some calculations reduce to

$$\widetilde{V} = V - \eta', \tag{2.17}$$

$$\gamma = \frac{\eta'}{2} + \frac{\eta^2}{2} - 2V + d, \tag{2.18}$$

$$\frac{\eta\eta''}{2} - \frac{\eta'^2}{4} + \eta^2\eta' + \frac{\eta^4}{4} - 2V\eta^2 + d\eta^2 + c = 0, \tag{2.19}$$

with $c, d \in \mathbb{R}$. For a given $V(x)$, the new potential $\widetilde{V}(x)$ and $\gamma(x)$ are obtained from (2.17) and (2.18) once we find a solution $\eta(x)$ of (2.19), which can be gotten from the ansatz

$$\eta' = -\eta^2 + 2\beta\eta + 2\xi. \tag{2.20}$$

By plugging (2.20) into (2.19), after some calculations we get $\xi^2 \equiv c$ and

$$\beta'(x) + \beta^2(x) = 2[V(x) - \epsilon], \qquad \epsilon = (d + \xi)/2, \tag{2.21}$$

which is again a Riccati equation. We can work as well the related Schrödinger equation, which arises by substituting into (2.21) $\beta(x) = [\ln u(x)]'$:

$$-\frac{u''}{2} + Vu = \epsilon u. \tag{2.22}$$

If $c \neq 0, \xi$ takes the values $\pm\sqrt{c}$, and in this way we need to solve the Riccati equation (2.21) for two factorization energies $\epsilon_{1,2} = (d \pm \sqrt{c})/2$. Then one constructs algebraically a common solution $\eta(x)$ of the corresponding pair of equations (2.20). On the other hand, if $c = 0$ one has to solve first the Riccati equation (2.21) for $\epsilon = d/2$ and to find after the general solution of the Bernoulli equation resulting for $\eta(x)$ (see (2.20)). There is a clear difference between the situation with real factorization constants ($c > 0$) and the complex case ($c < 0$), suggesting to classify the solutions $\eta(x)$ based on the sign of c, which is next elaborated [30].

2.2.1. Real case ($c > 0$). Here we have $\epsilon_{1,2} \in \mathbb{R}, \epsilon_1 \neq \epsilon_2$, the corresponding Riccati solutions of (2.21) being denoted by $\beta_{1,2}(x)$. The resulting formula for $\eta(x)$, expressed either in terms of $\beta_{1,2}(x)$ or of the corresponding Schrödinger seed solutions $u_{1,2}(x)$ becomes

$$\eta(x) = -\frac{2(\epsilon_1 - \epsilon_2)}{\beta_1(x) - \beta_2(x)} = \frac{2(\epsilon_1 - \epsilon_2)u_1u_2}{W(u_1, u_2)} = \frac{W'(u_1, u_2)}{W(u_1, u_2)}, \tag{2.23}$$

where $W(f, g) = fg' - gf'$ is the Wronskian of f and g. It is clear from equations (2.17) and (2.23) that the new potential $\widetilde{V}(x)$ has no new singularities in (x_l, x_r) if $W(u_1, u_2)$ is nodeless there.

The spectrum of $\widetilde{H}$ depends on whether or not its two 'mathematical' eigenfunctions $\widetilde{\psi}_{\epsilon_{1,2}}$ associated with $\epsilon_{1,2}$ which belong as well to the kernel of B can be normalized, namely

$$B\widetilde{\psi}_{\epsilon_{1,2}} = 0, \qquad \widetilde{H}\widetilde{\psi}_{\epsilon_{1,2}} = \epsilon_{1,2}\widetilde{\psi}_{\epsilon_{1,2}}.$$

Their explicit expressions in terms of $u_{1,2}$ are

$$\widetilde{\psi}_{\epsilon_1} \propto \frac{\eta}{u_1} \propto \frac{u_2}{W(u_1, u_2)}, \qquad \widetilde{\psi}_{\epsilon_2} \propto \frac{\eta}{u_2} \propto \frac{u_1}{W(u_1, u_2)}. \tag{2.24}$$

If both of them can be normalized, we then arrive at the maximal set of eigenfunctions of $\widetilde{H}$:

$$\left\{ \widetilde{\psi}_{\epsilon_1}, \widetilde{\psi}_{\epsilon_2}, \widetilde{\psi}_n = \frac{B^\dagger \psi_n}{\sqrt{(E_n - \epsilon_1)(E_n - \epsilon_2)}} \right\}. \tag{2.25}$$

Among the several spectral modifications which can be achieved through the real second-order SUSY QM, some cases are worth mentioning [15, 18].

(a) *Deleting two neighbour levels.* For $\epsilon_2 = E_{i-1}, \epsilon_1 = E_i, u_2 = \psi_{i-1}, u_1 = \psi_i$, it turns out that the Wronskian is nodeless and $\widetilde{\psi}_{\epsilon_1}, \widetilde{\psi}_{\epsilon_2}$ are non-normalizable. Thus, $\text{Sp}(\widetilde{H}) = \{E_0, \dots, E_{i-2}, E_{i+1}, \dots\}$, i.e., the levels E_{i-1}, E_i were 'deleted' for generating $\widetilde{V}(x)$.

(b) *Creating two new levels.* For $E_{i-1} < \epsilon_2 < \epsilon_1 < E_i, i = 1, 2, \dots$, by taking u_2, u_1 with $i+1, i$ nodes respectively the Wronskian becomes nodeless, $\widetilde{\psi}_{\epsilon_1}, \widetilde{\psi}_{\epsilon_2}$ are normalizable and $\text{Sp}(\widetilde{H}) = \{E_0, \dots, E_{i-1}, \epsilon_2, \epsilon_1, E_i, \dots\}$.

(c) *Isospectral transformations.* They appear as a limit of the case in which two new levels are created for $E_{i-1} < \epsilon_2 < \epsilon_1 < E_i$, when $u_{1,2}$ satisfy either $u_{1,2}(x_l) = 0$ or $u_{1,2}(x_r) = 0$. In this case the Wronskian vanishes at x_l or x_r, and $\widetilde{\psi}_{\epsilon_1}, \widetilde{\psi}_{\epsilon_2}$ cease to be normalizable so that $\text{Sp}(\widetilde{H}) = \text{Sp}(H)$.

2.2.2. Complex case ($c < 0$) *[31].* Now $\epsilon \equiv \epsilon_1 \in \mathbb{C}, \epsilon_2 = \bar{\epsilon}$, and since we look for $\widetilde{V}(x)$ real, it must be taken $\beta(x) \equiv \beta_1 = \bar{\beta}_2(x)$. Hence, the real solution $\eta(x)$ of equation (2.19) generated from the complex one $\beta(x)$ of (2.21) becomes

$$\eta(x) = -\frac{2\text{Im}(\epsilon)}{\text{Im}[\beta(x)]} = \frac{w'(x)}{w(x)}, \qquad w(x) = \frac{W(u, \bar{u})}{2(\epsilon - \bar{\epsilon})}. \tag{2.26}$$

Note that $w(x)$ must be nodeless for $x \in (x_l, x_r)$ to avoid new singularities in $\widetilde{V}(x)$. Since $w(x)$ is non-decreasing monotonic ($w'(x) = |u(x)|^2$), a sufficient condition ensuring the lack of zeros is

$$\lim_{x\to x_l} u(x) = 0 \qquad \text{or} \qquad \lim_{x\to x_r} u(x) = 0. \tag{2.27}$$

For transformation functions obeying (2.27), $\widetilde{V}(x)$ is a real potential isospectral to $V(x)$.

2.2.3. Confluent case ($c = 0$) *[32, 33].* We get now $\xi = 0, \epsilon \equiv \epsilon_1 = \epsilon_2 \in \mathbb{R}$; let us take a Riccati solution $\beta(x)$ to (2.21) for the given ϵ. Thus, the general solution for the Bernoulli equation resulting of (2.20) reads

$$\eta(x) = \frac{e^{2\int \beta(x)\,\mathrm{d}x}}{\widetilde{w}_0 + \int e^{2\int \beta(x)\,\mathrm{d}x}\,\mathrm{d}x} = \frac{w'(x)}{w(x)}, \tag{2.28}$$

$$w(x) = \widetilde{w}_0 + \int e^{2\int \beta(x)\,\mathrm{d}x}\,\mathrm{d}x = w_0 + \int_{x_0}^{x} [u(y)]^2\,\mathrm{d}y, \tag{2.29}$$

where x_0 is a fixed point in $[x_l, x_r]$. Once again, $w(x)$ must be nodeless in order that $\widetilde{V}(x)$ has no singularities in (x_l, x_r). Since $w(x)$ is non-decreasing monotonic ($w'(x) = [u(x)]^2$), the simplest choice ensuring a nodeless $w(x)$ is to take $u(x)$ satisfying either

$$\lim_{x\to x_l} u(x) = 0, \qquad I_- = \int_{x_l}^{x_0} [u(y)]^2\,\mathrm{d}y < \infty \tag{2.30}$$

or

$$\lim_{x\to x_r} u(x) = 0, \qquad I_+ = \int_{x_0}^{x_r} [u(y)]^2\,\mathrm{d}y < \infty. \tag{2.31}$$

In both cases it is possible to find a w_0-domain for which $w(x)$ is nodeless. The spectrum of $\widetilde{H}$ depends on the normalizability of the eigenfunction $\widetilde{\psi}_\epsilon$ of $\widetilde{H}$ associated with ϵ belonging as well to the kernel of B, with explicit expression given by

$$\widetilde{\psi}_\epsilon(x) \propto \frac{\eta(x)}{u(x)} \propto \frac{u(x)}{w(x)}.$$

If it can be normalized, then the maximal set of eigenfunctions of $\widetilde{H}$ becomes

$$\left\{\widetilde{\psi}_\epsilon(x), \widetilde{\psi}_n(x) = \frac{B^\dagger\psi_n(x)}{E_n - \epsilon}\right\}. \tag{2.32}$$

Note that, for $\epsilon > E_0, \epsilon \neq E_m, m = 1, 2, \ldots$ there exist solutions u satisfying (2.30) or (2.31) such that $\widetilde{\psi}_\epsilon$ is normalizable, i.e., the confluent second-order SUSY QM allows to embed a *single* level above the ground state of H. Moreover, since the physical eigenfunctions of H satisfy both (2.30) and (2.31), they are also appropriate for implementing the confluent algorithm. Let us remark that, apparently, the first authors who realized that through the confluent SUSY QM it is possible to modify the excited part of the spectrum were Baye and collaborators [34, 35]. We thank one of the referees of this paper for this information.

3. Trigonometric Pöschl–Teller potentials and their SUSY partners

Let us apply the previous techniques to the trigonometric Pöschl–Teller potentials [16, 27, 29]:

$$V(x) = \frac{(\lambda-1)\lambda}{2\sin^2(x)} + \frac{(\nu-1)\nu}{2\cos^2(x)}, \qquad \lambda, \nu > 1. \tag{3.1}$$

Note that, for $1/2 < \lambda = \nu < 1$, the $V(x)$ of (3.1) is known as Scarf potential [9, 28]. The SUSY transformations for that periodic potential have been recently implemented [24].

Along the paper the general solution of the Schrödinger equation $Hu(x) = \epsilon u(x)$ for any positive value of the energy parameter ϵ will be extensively used, which reads

$$\begin{aligned} u(x) = \sin^\lambda(x)\cos^\nu(x)\Bigg\{&A\,{}_2F_1\left[\frac{\mu}{2} + \sqrt{\frac{\epsilon}{2}}, \frac{\mu}{2} - \sqrt{\frac{\epsilon}{2}}; \lambda + \frac{1}{2}; \sin^2(x)\right] \\ &+ B\sin^{1-2\lambda}(x)\,{}_2F_1\left[\frac{1+\nu-\lambda}{2} + \sqrt{\frac{\epsilon}{2}}, \frac{1+\nu-\lambda}{2} - \sqrt{\frac{\epsilon}{2}}; \frac{3}{2} - \lambda; \sin^2(x)\right]\Bigg\}, \end{aligned} \tag{3.2}$$

where $\mu = \lambda + \nu$. We can find now the eigenfunctions $\psi_n(x)$ of H, which satisfy the boundary conditions $\psi_n(0) = \psi_n(\pi/2) = 0$. Since $\psi_n(0) = 0$, it turns out that $B = 0$. Moreover, for arbitrary $\epsilon > 0$ the hypergeometric function involved in the remaining term diverges when $x \to \pi/2$ stronger than the vanishing behaviour induced by $\cos^\nu(x)$. In order to avoid this divergence so that $\psi_n(\pi/2) = 0$, one of the two first parameters of the corresponding hypergeometric function has to be a negative integer, namely:

$$\frac{\mu}{2} \pm \sqrt{\frac{E_n}{2}} = -n \quad \Rightarrow \quad E_n = \frac{(\mu+2n)^2}{2}, \qquad n = 0, 1, 2, \ldots \tag{3.3}$$

By using the normalization condition it turns out that the eigenfunctions of H are

$$\psi_n(x) = \sqrt{\frac{2(\mu+2n)n!\Gamma(\mu+n)\left(\lambda+\frac{1}{2}\right)_n}{\left(\nu+\frac{1}{2}\right)_n\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma^3\left(\nu+\frac{1}{2}\right)}}\sin^\lambda(x)\cos^\nu(x)\,{}_2F_1\left[-n, n+\mu; \lambda+\frac{1}{2}; \sin^2(x)\right]. \tag{3.4}$$

For implementing later the SUSY transformations, it is important to know the number of zeros of the Schrödinger seed solution which is going to be employed. These nodes depend on ϵ, A, B (see expression (3.2)). To determine that dependence, let us compare the asymptotic behaviour of $u(x)$ for $x \to 0, \pi/2$. Indeed:

$$u(x) \underset{x\to 0}{\sim} B\sin^{1-\lambda}(x), \qquad u(x) \underset{x\to\frac{\pi}{2}}{\sim} (Aa + Bb)\cos^{1-\nu}(x), \tag{3.5}$$

$$a = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma\left(\frac{\mu}{2}+\sqrt{\frac{\epsilon}{2}}\right)\Gamma\left(\frac{\mu}{2}-\sqrt{\frac{\epsilon}{2}}\right)}, \qquad b = \frac{\Gamma\left(\frac{3}{2}-\lambda\right)\Gamma\left(\nu-\frac{1}{2}\right)}{\Gamma\left(\frac{1+\nu-\lambda}{2}+\sqrt{\frac{\epsilon}{2}}\right)\Gamma\left(\frac{1+\nu-\lambda}{2}-\sqrt{\frac{\epsilon}{2}}\right)}.$$

By asking that $u(x) > 0$ when $x \sim 0$, it turns out that $B > 0$. Without loosing generality let us take $B = 1$ and $A = -b/a + q$. Since for $\epsilon < E_0$ $u(x)$ just can have either one or zero nodes in $(0, \pi/2)$, thus it will have one if $q < 0$ while it will be nodeless if $q > 0$. For $E_0 < \epsilon < E_1$, $u(x)$ will have either two zeros for $q < 0$ or just one for $q > 0$. In general, for $E_{i-1} < \epsilon < E_i$, $u(x)$ will have either $i + 1$ nodes for $q < 0$ or i ones for $q > 0$.

Note that the trigonometric Pöschl–Teller potentials, and the corresponding Hamiltonians, are invariant under the transformation $x \to \pi/2 - x, \lambda \to \nu, \nu \to \lambda$. Its action onto the Schrödinger solution (3.2), with a given ϵ and specific values of the parameters (A, B), produces another solution with different parameters $(A\alpha_1 + B\beta_1, A\alpha_2 + B\beta_2)$, where

$$\alpha_1 = -\left(\frac{2\nu-1}{2\lambda-1}\right)b, \qquad \alpha_2 = \left(\frac{2\nu-1}{2\lambda-1}\right)a,$$

$$\beta_1 = \frac{\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(1-\frac{\mu}{2}+\sqrt{\frac{\epsilon}{2}}\right)\Gamma\left(1-\frac{\mu}{2}-\sqrt{\frac{\epsilon}{2}}\right)}, \qquad \beta_2 = \frac{\Gamma\left(\lambda-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-\nu\right)}{\Gamma\left(\frac{1+\lambda-\nu}{2}+\sqrt{\frac{\epsilon}{2}}\right)\Gamma\left(\frac{1+\lambda-\nu}{2}-\sqrt{\frac{\epsilon}{2}}\right)}.$$

This result will be used below to diminish the number of discussed SUSY transformations.

3.1. First-order SUSY partners

Let us classify the first-order SUSY partners according to the changes induced on the initial spectrum. Three different cases have been identified [16].

(a) *Deleting the initial ground state.* Let us choose $\epsilon = E_0$ and as seed the ground-state eigenfunction of H,

$$u(x) = \psi_0(x) \propto \sin^{\lambda}(x)\cos^{\nu}(x). \tag{3.6}$$

The SUSY partner potential of $V(x)$ becomes

$$\widetilde{V}(x) = \frac{\lambda(\lambda+1)}{2\sin^2(x)} + \frac{\nu(\nu+1)}{2\cos^2(x)}, \qquad \lambda, \nu > 1. \tag{3.7}$$

Since $\widetilde{\psi}_\epsilon(x) \propto 1/\psi_0(x)$ diverges at $x = 0, \pi/2$, the eigenvalues of $\widetilde{H}$ are given by (3.3) just with $n = 1, 2, \ldots$, i.e., we have 'deleted' the ground-state energy of H to generate $\widetilde{V}(x)$.

The previous SUSY partner potential $\widetilde{V}(x)$ can be obtained of the initial one through the change $\lambda \to \lambda + 1, \nu \to \nu + 1$, a property which is nowadays called shape invariance [9]. The fact that the singularities at $x = 0, \pi/2$ are reinforced, increasing by one both parameters λ, ν, has to do with the vanishing at those points of the employed seed solution. This behaviour is identical to that observed at the origin for the singular term of the SUSY partners of effective radial potentials [6].

As an illustration, the potentials $\widetilde{V}(x)$ and $V(x)$ for $\lambda = 3, \nu = 4$ are drawn in dashed and in gray respectively in figure 1.

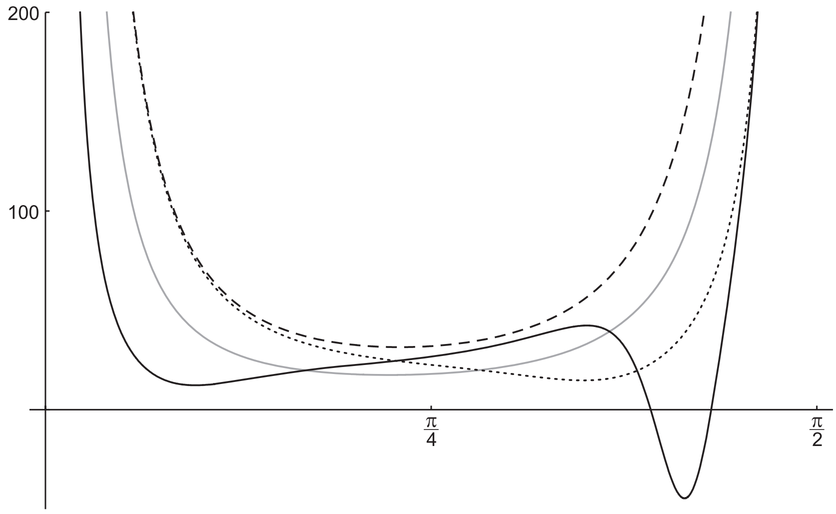

Figure 1. Trigonometric Pöschl–Teller potential for $\lambda = 3$, $\nu = 4$ (gray curve) and its first-order SUSY partners which arise from deleting the initial ground state $E_0 = 24.5$ (dashed curve), creating a new ground state at $\epsilon = 19$ (black continuous curve) and making an isospectral transformation with the same ϵ (dotted curve).

(b) *Creating a new ground state.* Let us take now $\epsilon < E_0$ and a nodeless seed solution $u(x)$ given by (3.2) with $B = 1$, $A = -b/a + q, q > 0$. Since $u(x) \to \infty$ as $x \to 0, \pi/2$, then $\widetilde{\psi}_\epsilon(0) = \widetilde{\psi}_\epsilon(\pi/2) = 0$, i.e., $\widetilde{\psi}_\epsilon(x)$ is a new eigenfunction of $\widetilde{H}$ with eigenvalue ϵ. Note that $\text{Sp}(\widetilde{H}) = \{\epsilon, E_n, n = 0, 1, \ldots\}$, namely, a new level has been 'created' at ϵ for $\widetilde{H}$. The singularities induced by $u(x)$ on $\widetilde{V}(x)$ at $x = 0, \pi/2$ are managed by defining

$$u(x) = \sin^{1-\lambda}(x)\cos^{1-\nu}(x)v(x), \tag{3.8}$$

where $v(x)$ is a nodeless bounded function in $[0, \pi/2]$. Thus we get

$$\widetilde{V}(x) = \frac{(\lambda-2)(\lambda-1)}{2\sin^2(x)} + \frac{(\nu-2)(\nu-1)}{2\cos^2(x)} - [\ln v(x)]'', \qquad \lambda, \nu > 2. \tag{3.9}$$

Note that now the singularities at $x = 0, \pi/2$ are weakened, decreasing by one both parameters λ, ν. This is due to the divergence at both points of the employed seed solution, which once again is similar to the behaviour at the origin for the singular term of the SUSY partners of effective radial potentials [3, 6, 10].

An example of the potential (3.9) for $\lambda = 3$, $\nu = 4$ is given by the black continuous curve of figure 1.

(c) *Isospectral potentials.* They appear from the transformations creating a new level at $\epsilon < E_0$ in the limit when $u(x)$ vanishes at one of the ends of the x-domain so that $\widetilde{\psi}_\epsilon(x)$ is not longer an eigenstate of $\widetilde{H}$. In our example, two appropriate seeds are available, given by (3.2) with $A = 1$, $B = 0$ or $A = -b/a$, $B = 1$. In the first case $u(0) = 0$, and the corresponding divergence induced on $\widetilde{V}(x)$ can be handled by taking:

$$u(x) = \sin^{\lambda}(x)\cos^{1-\nu}(x)v(x), \tag{3.10}$$

$v(x)$ being nodeless bounded in $[0, \pi/2]$. With this choice it turns out that

$$\widetilde{V}(x) = \frac{\lambda(\lambda+1)}{2\sin^2(x)} + \frac{(\nu-2)(\nu-1)}{2\cos^2(x)} - [\ln v(x)]'', \qquad \lambda > 1, \qquad \nu > 2. \tag{3.11}$$

Since $|\widetilde{\psi}_\epsilon(x)| \to \infty$ when $x \to 0$, then $\epsilon \notin \mathrm{Sp}(\widetilde{H})$ and therefore $\widetilde{H}$ is isospectral to H. Note the opposite changes of λ, ν suffered by the SUSY partner potentials $\widetilde{V}(x)$: the parameter λ (ν) is increased (decreased) by one since the seed solution vanishes (diverges) at $x = 0$ ($x = \pi/2$). Once again this is similar to the modifications induced by SUSY on the term singular at the origin of effective radial potentials [3, 6, 10].

The potential (3.11) for $\lambda = 3, \nu = 4$ is illustrated by the dotted curve of figure 1. On the other hand, the second seed solution which satisfies $u(\pi/2) = 0$ is obtained by changing $x \to \pi/2 - x, \lambda \to \nu, \nu \to \lambda$ in (3.10). The corresponding isospectral SUSY partner potential arises from the same transformation applied to (3.11).

3.2. *Second-order SUSY partners*

Let us explore the spectral modifications which can be induced in the three cases of the classification of section 2 (a partial study is found in [29]). Our results suggest a rule which will be observed for the changes induced on the parameters λ, ν characterizing the singularities at $x = 0, \pi/2$ in the real and complex cases: if both seeds vanish (diverge) at $x = 0$, then each one will increase (decrease) by one the parameter λ so that at the end the coefficient of the divergent term of $\widetilde{V}(x)$ is obtained by making $\lambda \to \lambda + 2$ ($\lambda \to \lambda - 2$). On the other hand, if one solution vanishes while the other one diverges at $x = 0$, then the corresponding singular term of $\widetilde{V}(x)$ will be the same as for $V(x)$ (unchanged λ). Something similar happens for the parameter ν characterizing the singularity at $x = \pi/2$. This behaviour is also seen for the singularity at the origin of the SUSY partners of effective radial potentials [6].

3.2.1. Real case. For $\epsilon_{1,2} \in \mathbb{R}$ several possibilities of modifying Sp(*H*) are available.

(a) *Deleting two neighbour levels.* Let us take $\epsilon_1 = E_i, \epsilon_2 = E_{i-1}, u_1(x) = \psi_i(x), u_2(x) = \psi_{i-1}(x)$ (see equation (3.4)). It is straightforward to show that

$$W(u_1, u_2) \propto \sin^{2\lambda+1}(x)\cos^{2\nu+1}(x)\mathcal{W}, \tag{3.12}$$

where

$$\mathcal{W} = \frac{W\left\{{}_2F_1\left[-i, i+\mu; \lambda + \frac{1}{2}; \sin^2(x)\right], {}_2F_1\left[-i+1, i-1+\mu; \lambda+\frac{1}{2}; \sin^2(x)\right]\right\}}{\sin(x)\cos(x)} \tag{3.13}$$

is a nodeless bounded function in $[0, \pi/2]$. The second-order SUSY partners of $V(x)$ become

$$\widetilde{V}(x) = \frac{(\lambda+1)(\lambda+2)}{2\sin^2(x)} + \frac{(\nu+1)(\nu+2)}{2\cos^2(x)} - (\ln\mathcal{W})'', \qquad \lambda, \nu > 1. \tag{3.14}$$

The two mathematical eigenfunctions $\widetilde{\psi}_{\epsilon_1} \propto u_2/W(u_1, u_2)$, $\widetilde{\psi}_{\epsilon_2} \propto u_1/W(u_1, u_2)$ of $\widetilde{H}$ associated with $\epsilon_1 = E_i, \epsilon_2 = E_{i-1}$ do not obey anymore the boundary conditions to be physical eigenfunctions of $\widetilde{H}$ since

$$\lim_{x\to 0, \frac{\pi}{2}} |\widetilde{\psi}_{\epsilon_{1,2}}(x)| = \infty.$$

Thus, $\mathrm{Sp}(\widetilde{H}) = \{E_0, \ldots E_{i-2}, E_{i+1}, \ldots\}$.

A plot of the potential (3.14) for $\lambda = 5, \nu = 8$, generated by deleting the levels $E_2 = 144.5$, $E_3 = 180.5$, is shown in dashed in figure 2, while the initial one is drawn in gray. Note the stronger intensities of the singularities at $x = 0, \pi/2$ of $\widetilde{V}(x)$ with respect to the corresponding ones of $V(x)$ (compare the potentials (3.1) and (3.14)).

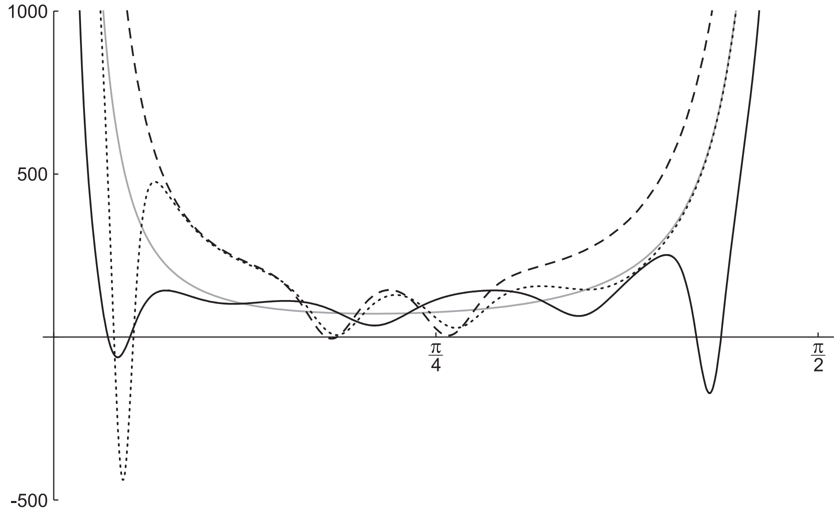

Figure 2. Trigonometric Pöschl–Teller potential for $\lambda = 5$, $\nu = 8$ (gray curve) and its second-order SUSY partners (real case) which arise by deleting the levels $E_2 = 144.5$, $E_3 = 180.5$ (dashed curve), creating two new eigenvalues at $\epsilon_1 = 128$, $\epsilon_2 = 115.52$ (black continuous curve), and moving the energy $E_2 = 144.5$ up to $\epsilon_1 = 169.28$ (dotted curve).

(b) *Creating two new levels.* Let us choose now $E_{i-1} < \epsilon_2 < \epsilon_1 < E_i$, and the corresponding seed solutions as given by (3.2) with $B_{1,2} = 1$, $A_{1,2} = -b_{1,2}/a_{1,2} + q_{1,2}$, $q_2 < 0$, $q_1 > 0$, i.e., u_2 and u_1 have $i + 1$ and i nodes respectively, making the Wronskian nodeless. In order to include the case when $\epsilon_2 < \epsilon_1 < E_0$, let us assume that $i = 0, 1, 2, \ldots$, where we have introduced the formal fictitious level $E_{-1} \equiv -\infty$. It is important to 'isolate' the divergent behaviour of the u solutions for $x \to 0$ and $x \to \pi/2$ (see equation (3.5)) by taking

$$u_{1,2}(x) = \sin^{1-\lambda}(x)\cos^{1-\nu}(x)v_{1,2}(x), \tag{3.15}$$

$v_{1,2}(x)$ being bounded for $x \in [0, \pi/2]$, $v_{1,2}(0) \neq 0$, $v_{1,2}(\pi/2) \neq 0$. Since the second term in the Taylor series expansion of $v_{1,2}(x)$ is proportional to $\sin^2(x)$, it turns out that $v'_{1,2}(x)$ tend to zero as $\sin(x)$ for $x \to 0$ and as $\cos(x)$ for $x \to \pi/2$. A simple calculation leads to

$$W(u_1, u_2) = \sin^{3-2\lambda}(x)\cos^{3-2\nu}(x)\mathcal{W}, \tag{3.16}$$

where $\mathcal{W} = W(v_1, v_2)/[\sin(x)\cos(x)]$ is nodeless bounded in $[0, \pi/2]$. The second-order SUSY partners of the Pöschl–Teller potential (3.1) are now:

$$\widetilde{V}(x) = \frac{(\lambda-3)(\lambda-2)}{2\sin^2(x)} + \frac{(\nu-3)(\nu-2)}{2\cos^2(x)} - (\ln\mathcal{W})'', \quad \lambda, \nu > 3. \tag{3.17}$$

Since

$$\lim_{x\to 0,\frac{\pi}{2}} \widetilde{\psi}_{\epsilon_{1,2}}(x) = 0,$$

then $\mathrm{Sp}(\widetilde{H}) = \{E_0, \ldots, E_{i-1}, \epsilon_2, \epsilon_1, E_i, \ldots\}$, i.e., two new levels have been created between a pair of neighbour ones of H to generate $\widetilde{V}(x)$.

A plot of the potentials (3.17) for $\lambda = 5$, $\nu = 8$, generated by creating the two new levels $\epsilon_1 = 128$, $\epsilon_2 = 115.52$, is given by the black continuous curve of figure 2. Observe the weaker intensities of the singularities at $x = 0, \pi/2$ of $\widetilde{V}(x)$ compared with those of the initial potential (3.1).

(c) *Isospectral transformations.* They arise from those which create two new levels (see case *(b)*) in the limit when each seed vanishes at one of the ends of the x-domain. By simplicity, let us choose $u_{1,2}$ as given in (3.2) with $B_{1,2} = 0$, $A_{1,2} = 1$ so that $u_{1,2}(0) = 0$. Since $|u_{1,2}(x)| \to \infty$ when $x \to \pi/2$, it is convenient to express:

$$u_{1,2}(x) = \sin^{\lambda}(x)\cos^{1-\nu}(x)v_{1,2}(x), \tag{3.18}$$

$v_{1,2}(x)$ being bounded in $[0, \pi/2]$, $v_{1,2}(0) \neq 0$, $v_{1,2}(\pi/2) \neq 0$. Once again, it turns out that:

$$W(u_1, u_2) = \sin^{2\lambda+1}(x)\cos^{3-2\nu}(x)\mathcal{W}, \tag{3.19}$$

where $\mathcal{W} = W(v_1, v_2)/[\sin(x)\cos(x)]$ is nodeless bounded in $[0, \pi/2]$. The second-order SUSY partners of the Pöschl–Teller potential are now

$$\widetilde{V}(x) = \frac{(\lambda+1)(\lambda+2)}{2\sin^2(x)} + \frac{(\nu-3)(\nu-2)}{2\cos^2(x)} - (\ln\mathcal{W})'', \qquad \lambda > 1, \qquad \nu > 3. \tag{3.20}$$

Note that

$$\lim_{x\to 0}|\widetilde{\psi}_{\epsilon_{1,2}}(x)| = \infty, \qquad \lim_{x\to\frac{\pi}{2}}\widetilde{\psi}_{\epsilon_{1,2}}(x) = 0.$$

This implies that $\epsilon_{1,2} \notin \mathrm{Sp}(\widetilde{H})$, meaning that $\widetilde{V}(x)$ is strictly isospectral to $V(x)$. Note that a similar procedure for $u_{1,2}$ satisfying $u_{1,2}(\pi/2) = 0$ can be applied. The corresponding seed solutions and isospectral SUSY partner potentials are obtained by changing $x \to \pi/2 - x$, $\lambda \to \nu$, $\nu \to \lambda$ in equations (3.18)–(3.20).

(d) *Creating a new level.* It appears from case (b) when one of the $i+1$ nodes of u_2 goes either to 0 or to $\pi/2$. In the first case it is taken $B_2 = 0$, $A_2 = 1$, $B_1 = 1$, $A_1 = -b_1/a_1 + q_1$, $q_1 > 0$, so that $u_2(0) = 0$. In order to manage the singularity at $x = \pi/2$ induced by $u_{1,2}$ on $\widetilde{V}(x)$, it is convenient to write them as:

$$u_1(x) = \sin^{1-\lambda}(x)\cos^{1-\nu}(x)v_1(x), \qquad u_2(x) = \sin^{\lambda}(x)\cos^{1-\nu}(x)v_2(x), \tag{3.21}$$

$v_{1,2}(x)$ being bounded in $[0, \pi/2]$, $v_{1,2}(0) \neq 0$, $v_{1,2}(\pi/2) \neq 0$. It can be shown that

$$W(u_1, u_2) = \cos^{3-2\nu}(x)\mathcal{W}, \tag{3.22}$$

where $\mathcal{W} = W[\sin^{1-\lambda}(x)v_1(x), \sin^{\lambda}(x)v_2(x)]/\cos(x)$ is nodeless bounded for $x \in [0, \pi/2]$. The second-order SUSY partner potentials of $V(x)$ are

$$\widetilde{V}(x) = \frac{(\lambda-1)\lambda}{2\sin^2(x)} + \frac{(\nu-3)(\nu-2)}{2\cos^2(x)} - (\ln\mathcal{W})'', \qquad \lambda > 1, \qquad \nu > 3. \tag{3.23}$$

Since

$$\lim_{x\to 0,\frac{\pi}{2}}\widetilde{\psi}_{\epsilon_1}(x) = \lim_{x\to\frac{\pi}{2}}\widetilde{\psi}_{\epsilon_2}(x) = 0, \qquad \lim_{x\to 0}|\widetilde{\psi}_{\epsilon_2}(x)| = \infty,$$

thus $\mathrm{Sp}(\widetilde{H}) = \{E_0, \ldots, E_{i-1}, \epsilon_1, E_i, \ldots\}$, i.e., we have embedded a new level ϵ_1 in (E_{i-1}, E_i).

The second possibility for generating a new level, in which $u_2(\pi/2) = 0$, can be obtained through the changes $x \to \pi/2 - x$, $\lambda \to \nu$, $\nu \to \lambda$ in formulae (3.21)–(3.23).

(e) *Moving an arbitrary level.* This can be achieved in the first place by taking the factorization energies as $E_{i-1} = \epsilon_2 < \epsilon_1 < E_i$ and the seeds in the way $u_2(x) = \psi_{i-1}(x), u_1(x)$ as given in (3.2) with $B_1 = 1, A_1 = -b_1/a_1 + q_1, q_1 > 0$ so that $u_1(x)$ has i nodes in $(0, \pi/2)$. It is convenient to factorize the null and divergent behaviour of the seed solutions $u_{1,2}(x)$ at $x = 0, \pi/2$ by expressing them as:

$$u_1(x) = \sin^{1-\lambda}(x)\cos^{1-\nu}(x)v_1(x), \qquad u_2(x) = \sin^{\lambda}(x)\cos^{\nu}(x)v_2(x), \tag{3.24}$$

where $v_{1,2}(x)$ are two bounded functions for $x \in [0, \pi/2]$, $v_{1,2}(0) \neq 0$, $v_{1,2}(\pi/2) \neq 0$. It turns out that $W(u_1, u_2)$ is nodeless bounded for $x \in [0, \pi/2]$. Moreover:

$$\lim_{x\to 0,\frac{\pi}{2}} \widetilde{\psi}_{\epsilon_1}(x) = 0, \qquad \lim_{x\to 0,\frac{\pi}{2}} |\widetilde{\psi}_{\epsilon_2}(x)| = \infty,$$

i.e., $\widetilde{\psi}_{\epsilon_1}(x)$ is an eigenfunction of $\widetilde{H}$ but $\widetilde{\psi}_{\epsilon_2}(x)$ is not. The second-order SUSY partners of $V(x)$ are given by

$$\widetilde{V}(x) = \frac{(\lambda-1)\lambda}{2\sin^2(x)} + \frac{(\nu-1)\nu}{2\cos^2(x)} - \{\ln[W(u_1, u_2)]\}'', \qquad \lambda, \nu > 1. \tag{3.25}$$

Since $\mathrm{Sp}(\widetilde{H}) = \{E_0, \dots, E_{i-2}, \epsilon_1, E_i, \dots\}$, we conclude that the level E_{i-1} has been *moved up* to achieve ϵ_1.

An example of the potentials (3.25) for $\lambda = 5$, $\nu = 8$ is plotted in figure 2 (dotted curve). The initial level $E_2 = 144.5$ has been moved up to achieve $\epsilon_1 = 169.28$. The 'intensities' of the singularities at $x = 0, \pi/2$ for $\widetilde{V}(x)$ remain the same as for the initial potential (3.1).

Another possibility is to take $E_{i-1} < \epsilon_2 < \epsilon_1 = E_i$, the corresponding seed solutions in the way $u_1(x) = \psi_i(x)$, the $u_2(x)$ of (3.2) with $A_2 = -b_2/a_2 + q_2, q_2 < 0$, i.e., $u_1(x)$ and $u_2(x)$ have i and $i+1$ nodes respectively for $x \in (0, \pi/2)$. It is convenient to express

$$u_1(x) = \sin^{\lambda}(x)\cos^{\nu}(x)v_1(x), \qquad u_2(x) = \sin^{1-\lambda}(x)\cos^{1-\nu}(x)v_2(x), \tag{3.26}$$

$v_{1,2}(x)$ being bounded for $x \in [0, \pi/2], v_{1,2}(0) \neq 0, v_{1,2}(\pi/2) \neq 0$. Once again, $W(u_1, u_2)$ is nodeless bounded for $x \in [0, \pi/2]$. Furthermore:

$$\lim_{x\to 0,\frac{\pi}{2}} |\widetilde{\psi}_{\epsilon_1}(x)| = \infty, \qquad \lim_{x\to 0,\frac{\pi}{2}} \widetilde{\psi}_{\epsilon_2}(x) = 0,$$

namely, $\widetilde{\psi}_{\epsilon_2}(x)$ is an eigenfunction of $\widetilde{H}$ while $\widetilde{\psi}_{\epsilon_1}(x)$ is not. The SUSY partner of $V(x)$ is given as well by (3.25), where now $\mathrm{Sp}(\widetilde{H}) = \{E_0, \dots, E_{i-1}, \epsilon_2, E_{i+1}, \dots\}$, meaning that the level E_i has been *moved down* to achieve ϵ_2.

(f) *Deleting an arbitrary level.* This is attained of the previous case in the limit when the nonphysical seed acquires one zero at $x = 0$ or $x = \pi/2$. For $E_{i-1} = \epsilon_2 < \epsilon_1 < E_i$ one possibility is to take $u_2(x) = \psi_{i-1}(x), u_1(x)$ as in (3.2) with $A_1 = 1, B_1 = 0$, so that $u_1(0) = 0$. Thus

$$u_1(x) = \sin^{\lambda}(x)\cos^{1-\nu}(x)v_1(x), \qquad u_2(x) = \sin^{\lambda}(x)\cos^{\nu}(x)v_2(x), \tag{3.27}$$

$v_{1,2}(x)$ being bounded for $x \in [0, \pi/2]$, $v_{1,2}(0) \neq 0$, $v_{1,2}(\pi/2) \neq 0$. It turns out that

$$W(u_1, u_2) = \sin^{2\lambda+1}(x)\mathcal{W}, \tag{3.28}$$

where $\mathcal{W} = W[\cos^{1-\nu}(x)v_1(x), \cos^{\nu}(x)v_2(x)]/\sin(x)$ is nodeless bounded for $x \in [0, \pi/2]$. Now we have

$$\lim_{x\to\frac{\pi}{2}} \widetilde{\psi}_{\epsilon_1}(x) = 0, \qquad \lim_{x\to 0} |\widetilde{\psi}_{\epsilon_1}(x)| = \lim_{x\to 0,\frac{\pi}{2}} |\widetilde{\psi}_{\epsilon_2}(x)| = \infty,$$

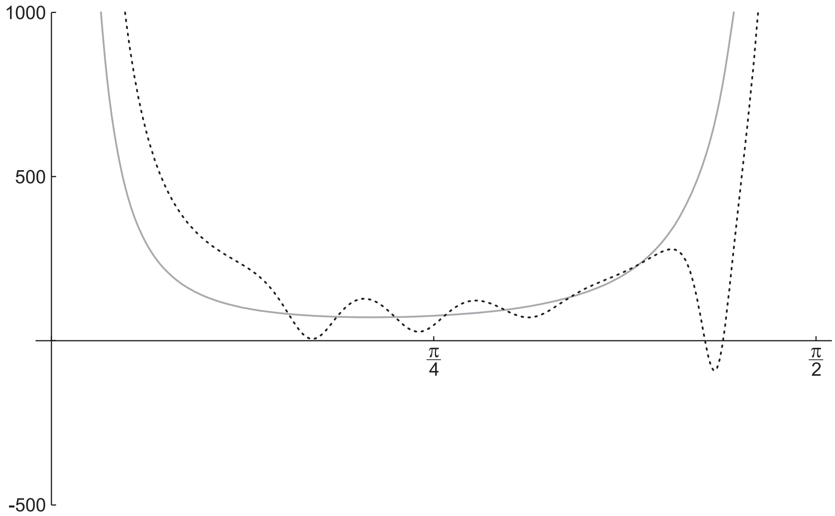

Figure 3. Trigonometric Pöschl–Teller potential for $\lambda = 5$, $\nu = 8$ (gray curve) and its second-order SUSY partner (complex case) which arises by using $\epsilon = 176.344 + 1.5\text{i}$ with a seed vanishing at the origin (dotted curve).

i.e., $\epsilon_{1,2} \notin \text{Sp}(\widetilde{H}) = \{E_0, \ldots, E_{i-2}, E_i, E_{i+1}, \ldots\}$. The SUSY partner potentials of $V(x)$ are given by

$$\widetilde{V}(x) = \frac{(\lambda+1)(\lambda+2)}{2\sin^2(x)} + \frac{(\nu-1)\nu}{2\cos^2(x)} - (\ln \mathcal{W})'', \qquad \lambda, \nu > 1. \tag{3.29}$$

It is seen that the level E_{i-1} has been deleted for generating $\widetilde{V}(x)$.
Another option for deleting the level E_{i-1} can be achieved by changing $x \to \pi/2 - x$, $\lambda \to \nu$, $\nu \to \lambda$ in equations (3.27)–(3.29).

3.2.2. Complex case. For $\epsilon \in \mathbb{C}$ the solution u given in (3.2) is still valid, and the condition (2.27) required to avoid the zeros in the Wronskian can be accomplished in two ways. In the first place we make $A = 1$, $B = 0$ and thus $u(0) = 0$ while $|u(x)| \to \infty$ as $x \to \pi/2$. The singularities induced on $\widetilde{V}(x)$ are handled by factorizing

$$u(x) = \sin^{\lambda}(x)\cos^{1-\nu}(x)v(x). \tag{3.30}$$

Therefore:

$$W(u, \bar{u}) = \sin^{2\lambda+1}(x)\cos^{3-2\nu}(x)\mathcal{W}, \tag{3.31}$$

$$\begin{aligned} \widetilde{V}(x) &= \frac{(\lambda+1)(\lambda+2)}{2\sin^2(x)} + \frac{(\nu-3)(\nu-2)}{2\cos^2(x)} - (\ln \mathcal{W})'', \qquad \lambda > 1, \qquad \nu > 3, \\ \mathcal{W} &= \frac{W(v, \bar{v})}{2(\epsilon - \bar{\epsilon})\sin(x)\cos(x)}. \end{aligned} \tag{3.32}$$

The potentials $\widetilde{V}(x)$ of (3.32) and the Pöschl–Teller initial one (3.1) are isospectral. Their plots for $\lambda = 5$, $\nu = 8$ are shown in figure 3, where the initial potential is drawn in gray while the dotted curve represents that of (3.32).

Note that the second solution satisfying $u(\pi/2) = 0, \lim_{x\to 0}|u(x)| \to \infty$, and the corresponding SUSY partner potential $\widetilde{V}(x)$, are obtained by changing $x \to \pi/2 - x, \lambda \to \nu, \nu \to \lambda$ in equations (3.30)–(3.32).

3.2.3. Confluent case. For $\epsilon = \epsilon_1 = \epsilon_2$, several possibilities of modifying the initial spectrum appear.

(a) *Creating a new level.* Let us choose $\mathbb{R} \ni \epsilon \neq E_i$, for which two seeds are available for implementing the confluent algorithm. The first one arises by taking $A = 1, B = 0$ in (3.2):

$$u(x) = \sin^\lambda(x)\cos^\nu(x)\,{}_2F_1\left(\frac{\mu}{2}+\sqrt{\frac{\epsilon}{2}}, \frac{\mu}{2}-\sqrt{\frac{\epsilon}{2}}; \lambda+\frac{1}{2}; \sin^2(x)\right) = \sin^\lambda(x)\cos^{1-\nu}(x)v(x), \tag{3.33}$$

$v(x)$ being bounded for $x \in [0, \pi/2], v(0) \neq 0, v(\pi/2) \neq 0$. The calculation of the integral of equation (2.29) with $x_0 = 0$ leads to

$$w(x) = w_0 + \sum_{m=0}^{\infty} \frac{\left(\frac{\mu}{2}+\sqrt{\frac{\epsilon}{2}}\right)_m\left(\frac{\mu}{2}-\sqrt{\frac{\epsilon}{2}}\right)_m \sin^{2\lambda+2m+1}(x)}{\left(\lambda+\frac{1}{2}\right)_m m!(2\lambda+2m+1)} \times {}_3F_2\left(\frac{1+\lambda-\nu}{2}-\sqrt{\frac{\epsilon}{2}}, \frac{1+\lambda-\nu}{2}+\sqrt{\frac{\epsilon}{2}}, \lambda+m+\frac{1}{2}; \lambda+\frac{1}{2}, \lambda+m+\frac{3}{2}; \sin^2(x)\right). \tag{3.34}$$

Note that $w(x)$ is nodeless in $[0, \pi/2]$ for $w_0 > 0$ while it will have one node for $w_0 < 0$. Let us choose a nodeless $w(x)$, as given in (3.34) with $w_0 > 0$. Its divergent behaviour for $x \to \pi/2$, being of kind $\cos^{3-2\nu}(x)$, will change the coefficient of the second term of the Pöschl–Teller potential (3.1), so it is convenient to factorize

$$w(x) = \cos^{3-2\nu}(x)\mathcal{W}(x), \tag{3.35}$$

$\mathcal{W}(x)$ being nodeless bounded for $x \in [0, \pi/2]$. The *confluent* second-order SUSY partner potentials of $V(x)$ become

$$\widetilde{V}(x) = \frac{(\lambda-1)\lambda}{2\sin^2(x)} + \frac{(\nu-3)(\nu-2)}{2\cos^2(x)} - \{\ln[\mathcal{W}(x)]\}'', \qquad \lambda > 1, \qquad \nu > 3. \tag{3.36}$$

Since $\widetilde{\psi}_\epsilon(x) \propto u(x)/w(x)$ satisfies

$$\lim_{x\to 0, \frac{\pi}{2}} \widetilde{\psi}_\epsilon(x) = 0, \tag{3.37}$$

then $\text{Sp}(\widetilde{H}) = \{\epsilon, E_n, n = 0, 1, \ldots\}, \epsilon \neq E_n$.
As an illustration, in figure 4 we have drawn in gray the initial potential for $\lambda = 5, \nu = 8$ and its SUSY partner (3.36) by the black continuous curve. The different intensities of the singularities for both potentials at $x = \pi/2$ are seen.
Note that the second seed, which is appropriate to implement the confluent algorithm, and the corresponding SUSY partner potential, are obtained by changing $x \to \pi/2 - x, \lambda \to \nu, \nu \to \lambda$ in equations (3.33)–(3.36).

(b) *Isospectral transformations.* They appear in several different ways, in the first place as two limits of the previous case when the eigenfunction of $\widetilde{H}$ associated with ϵ ceases to satisfy the right boundary conditions. This happens, e.g., if we take $u(x)$ as in (3.33) and the $w(x)$ of (3.34) with $w_0 = 0$. Besides the divergent behaviour of $w(x)$ as $x \to \pi/2$, it turns out that $w(x) \to 0$ as $\sin^{2\lambda+1}(x)$ when $x \to 0$, so that

$$w(x) = \sin^{2\lambda+1}(x)\cos^{3-2\nu}(x)\mathcal{W}(x), \tag{3.38}$$

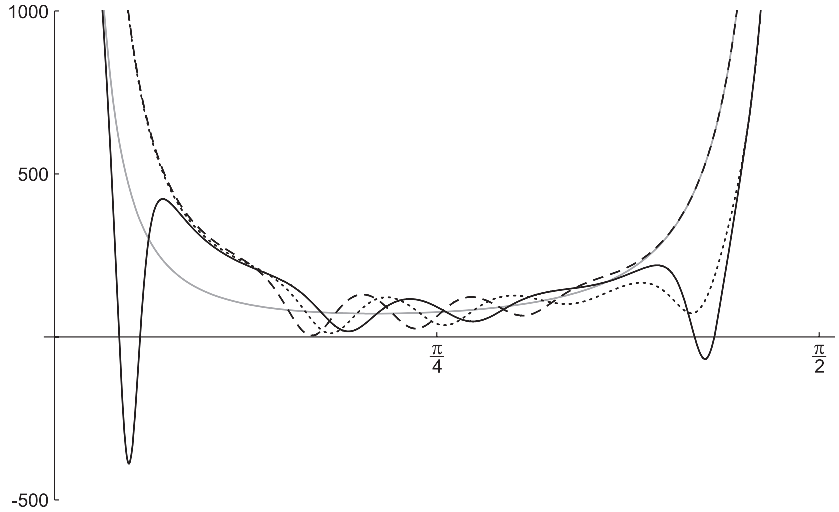

Figure 4. Trigonometric Pöschl–Teller potential for $\lambda = 5, \nu = 8$ (gray curve) and its second-order SUSY partners (confluent case) which arise from creating a new level at $\epsilon = 147.92$ (black continuous curve), making an isospectral transformation with $\epsilon = 162$ (dotted curve) and deleting the eigenvalue $E_3 = 180.5$ (dashed curve).

$\mathcal{W}(x)$ being nodeless bounded for $x \in [0, \pi/2]$. The SUSY partner potential of $V(x)$ is

$$\widetilde{V}(x) = \frac{(\lambda+1)(\lambda+2)}{2\sin^2(x)} + \frac{(\nu-3)(\nu-2)}{2\cos^2(x)} - \{\ln[\mathcal{W}(x)]\}'', \qquad \lambda > 1, \qquad \nu > 3. \tag{3.39}$$

Note that:

$$\lim_{x\to 0} |\widetilde{\psi}_\epsilon(x)| = \infty, \qquad \lim_{x\to \frac{\pi}{2}} \widetilde{\psi}_\epsilon(x) = 0, \tag{3.40}$$

i.e., $\epsilon \notin \mathrm{Sp}(\widetilde{H})$ and therefore $\widetilde{H}$ has the same spectrum as H.

An example of the potentials (3.39) for $\lambda = 5, \nu = 8, \epsilon = 162$ is shown in dotted in figure 4. It can be seen that the stronger intensity of the singularity at $x = 0$ of $\widetilde{V}(x)$, compared with $V(x)$, is 'compensated' by its lower value at $x = \pi/2$.

A second alternative to produce isospectral potentials consists in changing $x \to \pi/2 - x, \lambda \to \nu, \nu \to \lambda$ in equations (3.33)–(3.36) and taking $w_0 = 0$ in the resulting formulae. The corresponding SUSY partner potential is obtained by substituting $x \to \pi/2 - x, \lambda \to \nu, \nu \to \lambda$ into equations (3.38) and (3.39).

The third confluent isospectral transformation uses as seed physical eigenfunctions of H, i.e., $\epsilon = E_i, u(x) = \psi_i(x)$. The expression for $w(x)$ is obtained from (3.34) by realizing that the solution (3.33) is proportional to the eigenfunction (3.4) when $\epsilon \to E_i$,

$$\psi_i = c_i \lim_{\epsilon\to E_i} \sin^\lambda(x)\cos^\nu(x)\, {}_2F_1\left(\frac{\mu}{2} + \sqrt{\frac{\epsilon}{2}}, \frac{\mu}{2} - \sqrt{\frac{\epsilon}{2}}; \lambda + \frac{1}{2}; \sin^2(x)\right),$$
$$c_i = \left[\frac{2(\mu+2i)i!\,\Gamma(\mu+i)\left(\lambda+\frac{1}{2}\right)_i}{\left(\nu+\frac{1}{2}\right)_i\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma^3\left(\nu+\frac{1}{2}\right)}\right]^{\frac{1}{2}}. \tag{3.41}$$

Moreover, in this limit the infinite summation of (3.34) truncates at $m = i$, so that:

$$w(x) = w_0 + c_i^2 \sum_{m=0}^{i} \frac{(\mu + i)_m (-i)_m \sin^{2\lambda+2m+1}(x)}{\left(\lambda + \frac{1}{2}\right)_m m!(2\lambda + 2m + 1)} \times {}_3F_2\left(\frac{1}{2} - \nu - i, \frac{1}{2} + \lambda + i, \lambda + m + \frac{1}{2}; \lambda + \frac{1}{2}, \lambda + m + \frac{3}{2}; \sin^2(x)\right). \tag{3.42}$$

If $w_0 > 0$ or $w_0 < -1$, $w(x)$ is nodeless bounded for $x \in [0, \pi/2]$. Now there is no change in the intensities of the singularities at $x = 0, \pi/2$ for $\widetilde{V}(x)$, namely:

$$\widetilde{V}(x) = \frac{(\lambda - 1)\lambda}{2\sin^2(x)} + \frac{(\nu - 1)\nu}{2\cos^2(x)} - \{\ln[w(x)]\}'', \qquad \lambda, \nu > 1. \tag{3.43}$$

It turns out that

$$\lim_{x\to 0, \frac{\pi}{2}} \widetilde{\psi}_\epsilon(x) = 0, \tag{3.44}$$

i.e., $\epsilon = E_i \in \mathrm{Sp}(\widetilde{H})$ and thus H and $\widetilde{H}$ are isospectral.

(c) *Deleting an arbitrary level.* This case appears in the limits as $w_0 \to 0, -1$ of the isospectral transformations involving as seed the physical eigenfunction $\psi_i(x)$. For $w_0 \to 0$, $w(x) \sim \sin^{2\lambda+1}(x)$ when $x \to 0$ so that:

$$w(x) = \sin^{2\lambda+1}(x)\mathcal{W}(x), \tag{3.45}$$

where $\mathcal{W}(x)$ is nodeless bounded in $[0, \pi/2]$. Since

$$\lim_{x\to 0} |\widetilde{\psi}_\epsilon(x)| = \infty, \qquad \lim_{x\to \frac{\pi}{2}} \widetilde{\psi}_\epsilon(x) = 0, \tag{3.46}$$

then $\epsilon = E_i \notin \mathrm{Sp}(\widetilde{H}) = \{E_0, \ldots, E_{i-1}, E_{i+1}, \ldots\}$. The SUSY partner potential of $V(x)$ is

$$\widetilde{V}(x) = \frac{(\lambda + 1)(\lambda + 2)}{2\sin^2(x)} + \frac{(\nu - 1)\nu}{2\cos^2(x)} - \{\ln[\mathcal{W}(x)]\}'', \qquad \lambda, \nu > 1. \tag{3.47}$$

It is seen that we have deleted the level E_i to produce $\widetilde{V}(x)$.

An illustration of the potentials (3.47) for $\lambda = 5$, $\nu = 8$ is shown in dashed in figure 4. The deleted level is $E_3 = 180.5$, and the intensities of $V(x)$ and $\widetilde{V}(x)$ at $x = 0$ differ as predicted by equations (3.1) and (3.47).

The case when $w_0 \to -1$, which also leads to the deletion of the level E_i, can be achieved from equations (3.45) and (3.47) by the change $x \to \pi/2 - x$, $\lambda \to \nu$, $\nu \to \lambda$.

4. Conclusions

The supersymmetric quantum mechanics of first and second order have been used to generate new exactly solvable Hamiltonians departing from the trigonometric Pöschl–Teller potentials. Several interesting possibilities to modify the initial spectrum have been studied, and it has been shown that the deformations induced by the second-order algorithm can be non-standard, in the sense that the main spectral changes appear above the ground-state energy of the initial Hamiltonian. Specifically, we have shown that a pair of levels can be embedded between two neighbour initial ones. It has also been possible to delete two neighbour energies. Specially interesting is the possibility of embedding a single level at any arbitrary position. In addition, it is possible to move up or down a generic physical energy as well as to delete it. It is worth noting that some spectral modification can be achieved in several different ways.

For example, the strictly isospectral mappings can be obtained through the real, complex and confluent second-order transformations (see the potentials in (3.20), (3.32), (3.39) and (3.43)). However, if we want to produce an isospectral potential such that the coefficients of the singularities at $x = 0, \pi/2$ would be changed in a specific way, then the number of options becomes smaller. In particular, if the isospectral SUSY transformation is not going to modify the intensities of the two singularities at $x = 0, \pi/2$, then we will have to apply a confluent transformation involving as seed a physical eigenfunction of the trigonometric Pöschl–Teller Hamiltonian (see equation (3.43)). A similar discussion could be elaborated for the other cases having several possibilities to achieve the same final spectrum. Our general conclusion is that the supersymmetric quantum mechanics is a powerful mathematical tool for designing potentials with an arbitrarily prescribed spectrum.

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